

# Entanglement in two site Bose-Hubbard model

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## Abstract

In this paper, we are studying the decoherence and entanglement properties for the two site Bose-Hubbard model in presence of a non-linear damping. We apply the techniques of thermo field dynamics and then use Hartree-Fock approximation to solve the corresponding master equation. The expectation values of the approximated field is computed self-consistently. We solve this master equation for a small time  $t$  so that

we get the analytical solution, there by we compute the decoherence and entanglement properties of the two-mode bosonic system.

# 1 Introduction

In recent years, there is lot of interest generated in the study of entanglement properties of ultra cold atoms [1, 2, 3, 4, 5, 6]. In one such study, the single-site addressability in a two-dimensional optical lattice [8] has been demonstrated which could be a natural resource for applications of quantum information processing with neutral atoms. One of the popular model used to study the evolution of cold atoms and the BoseEinstein condensates in an optical lattice is Bose-Hubbard model [7]. In all the experimental demonstrations of ultra cold atoms, loss is an important role which gives rise to decoherence and in turn destroying the quantum correlations. In this paper, we examin the two-site BoseHubbard model to study the entanglement and decoherence properties of states under the action of non-linear damping. We consider the following master equation for density matrix  $\rho$  in a non-linear medium

$$\frac{\partial}{\partial t}\rho = \frac{i}{\hbar}[H, \rho] + \kappa \sum_{k=1}^K ([a_k b_k, \rho a_k^\dagger b_k^\dagger] + [a_k b_k \rho, a_k^\dagger b_k^\dagger]) \quad (1)$$

where  $a_k$  and  $b_k$  are bosonic annihilation operators,  $\kappa$  is a damping coefficient and  $H$  is the Hamiltonian for the Bose-Hubbard model which describes the optical lattice, discussed in the next section in detail . In ref [5, 6], the authors have studied the decoherence properties the Bose- Hubbard model with single mode damping. In this paper, we are studying the model in the presence of non-linear damping.

For solving this master equation we use the techniques of thermo field dynamics and thereby the Hartree-Fock approximation to convert two-site Bose-Hubbard model into a two-mode bosonic system. The two-site Bose-Hubbard model is used to study Josephson tunneling between two Bose-Einstein condensates. The expectation values of the approximated field is treated computed self-consistently. We solve this master equation for a small

time  $t$  so that we get the analytical solution, there by we compute the decoherence and entanglement properities of the two-mode bosonic system.

The thermo field dynamics (TFD)[9, 10, 11, 12, 13] is a finite temperature field theory. It is applied to many branches in high energy physics [10] and many-body systems [13]. The thermo field dynamics is used to solve the master equation by in presence of Kerr medium [14, 15] using disentanglement theorem for any arbitrary initial conditions. This formalism presented in ref [14, 15, 18, 19, 20] has two sailent features, first, solving of the master equation is reduced to solving a Schroedinger equation, thus all the techniques available to solve the Schroedinger equation are applicable here. Second, the thermal coherent state under the master equation evolution goes over to a thermal coherent state.

A brief description of TFD is given in appendix D. In TFD, the master equation is given by

$$\frac{\partial}{\partial t}|\rho(t)\rangle = -i\hat{H}|\rho\rangle \quad (2)$$

where  $|\rho\rangle$  is a vector in an extended Hilbert space  $\mathcal{H} \otimes \mathcal{H}^*$  and

$$-i\hat{H} = i(H - \tilde{H}) + L. \quad (3)$$

where  $H$  is the Hamiltonian in Hilbert space  $\mathcal{H}$ ,  $\tilde{H}$  is the Hamiltonian in Hilbert space  $\mathcal{H}^*$  and  $L$  is the Liouville term, as mentioned in appendix D . Thus,  $-i\hat{H}$  is tildian and the problem of solving master equation is reduced to solving a Schroedinger like equation namely eq (2). Then, the symmetry associated to the Hamiltonian such as  $su(1,1)$  are exploited to solve master equation.

In this paper, we follow this formalism to solve the master equation(1) for the two site Bose-Hubbard model in the presence of non-linear medium.

## 2 Two site Bose-Hubbard Model

The Hamiltonian of the Bose-Hubbard model describes the interaction of bosons on a optical lattice and is given by

$$\begin{aligned}
H = & \omega \sum_k (a_k^\dagger a_k + b_k^\dagger b_k) - J \sum_{\{k,l\}} (a_k^\dagger b_l + b_k^\dagger a_l) \\
& + \frac{U_a}{2} \sum_k a_k^\dagger a_k^\dagger a_k a_k + \frac{U_b}{2} \sum_k b_k^\dagger b_k^\dagger b_k b_k + \frac{U_{ab}}{2} \sum_{\{k,l\}} a_k^\dagger a_k b_l^\dagger b_l
\end{aligned} \tag{4}$$

with  $a_k$  and  $b_k$  bosonic annihilation operators referring to atoms in the internal states  $|N_1\rangle$  and  $|N_2\rangle$ , respectively, with one boson in the  $k$ th lattice site and  $K$  is the number of lattice sites  $\{k, l\}$  refers to the adjacent lattice points  $k$  and  $l$ . The interaction term  $J$  in the Hamiltonian describes the induced hopping between adjacent cells. The  $\omega$  is the frequency of the atom in the lattice. The on-site interactions of atoms are described by  $U_a$  and  $U_b$ , and a nearest-neighbour interaction by  $U_{ab}$ . For further details see ref [7]. To study the decoherence and the entanglement properties of Bose-Hubbard model, for simplicity, we consider the toy model, in which the Bose-Hubbard model is written for the two site interaction only. The master equation (1) for the two site Bose-Hubbard Hamiltonian  $H$  (given in (4)) is given by

$$\begin{aligned}
\frac{\partial}{\partial t} \rho = & -i\omega(a^\dagger a \rho - \rho a^\dagger a) - iJ(a^\dagger b \rho - \rho a^\dagger b) - iJ(b^\dagger a \rho - \rho b^\dagger a) \\
& + i\frac{U_a}{2}(a^\dagger a^\dagger a a \rho - \rho a^\dagger a^\dagger a a) + i\frac{U_b}{2}(b^\dagger b^\dagger b b \rho - \rho b^\dagger b^\dagger b b) \\
& - i\omega(b^\dagger b \rho - \rho b b^\dagger) + i\frac{U_{ab}}{2}(a^\dagger b^\dagger a b \rho - \rho a^\dagger b^\dagger a b) \\
& + \frac{\kappa}{2} (2ab\rho a^\dagger b^\dagger - a^\dagger b^\dagger a b \rho - \rho a^\dagger b^\dagger a b).
\end{aligned} \tag{5}$$

At first we consider the special case to solve this master equation with  $J = U_a = U_b = 0$  and  $U_{ab} = U$ , which corresponds to the Mott insulating phase. Then the master equation (5) reduces to

$$\begin{aligned}
\frac{\partial}{\partial t} \rho = & -i\omega(a^\dagger a \rho - \rho a^\dagger a) - i\omega(b^\dagger b \rho - \rho b b^\dagger) + i\frac{U}{2}(a^\dagger b^\dagger a b \rho - \rho a^\dagger b^\dagger a b) \\
& + \frac{\kappa}{2} (2ab\rho a^\dagger b^\dagger - a^\dagger b^\dagger a b \rho - \rho a^\dagger b^\dagger a b).
\end{aligned} \tag{6}$$

Now, we apply the thermo field dynamics techniques to convert the master equation (6) into a Schroedinger equation by applying  $|I\rangle$  from the right to the eq (6),

$$\frac{\partial}{\partial t}|\rho\rangle = -i\hat{H}|\rho\rangle, \quad (7)$$

where  $|\rho\rangle$  extended Hilbert space  $\mathcal{H} \otimes \mathcal{H}^*$  and the Hamiltonian is given by

$$\begin{aligned} -i\hat{H} = & -i\omega(a^\dagger a - \tilde{a}\tilde{a}^\dagger) - i\omega(b^\dagger b - \tilde{b}\tilde{b}^\dagger) + i\frac{U}{2}(a^\dagger b^\dagger ab - \tilde{a}\tilde{b}\tilde{a}^\dagger\tilde{b}^\dagger) \\ & + \frac{\kappa}{2}(2ab\tilde{a}\tilde{b} - a^\dagger b^\dagger ab - \tilde{a}\tilde{b}\tilde{a}^\dagger\tilde{b}^\dagger), \end{aligned} \quad (8)$$

here the  $a, b$  are the annihilation operators act on the  $\mathcal{H}$  and  $\tilde{a}$  and  $\tilde{b}$  are the annihilation operators act on the Hilbert space  $\mathcal{H}^*$  ( for detail see appendix D).

This master equation (8) is a non-linear equation, and in general it is difficult to get an analytical solution for this equation. A way out would be to apply the Hartree-Fock approximation and treat the approximated field self-consistently. In the case of thermo field dynamics, a selfconsistent theory using Hartree-Fock approximation is developed for the non linear master equations in presence of the non-linear medium in ref [20]. Applying, Hartree Fock approximation for each term in (8)as follows

$$\tilde{a}^\dagger\tilde{b}^\dagger\tilde{a}\tilde{b} = \tilde{a}^\dagger\tilde{b}^\dagger\langle\tilde{a}\tilde{b}\rangle, \quad ab\tilde{a}\tilde{b} = ab\langle\tilde{a}\tilde{b}\rangle, \quad \tilde{a}\tilde{b}ab = \tilde{a}\tilde{b}\langle ab\rangle, \quad (9)$$

with  $\langle ab\rangle = \langle\tilde{a}\tilde{b}\rangle = \Delta(t)$ , the Hamiltonian in (8) is decoupled into tildien and non tildian parts

$$-i\hat{H} = i(H_1 + H_2), \quad (10)$$

where

$$H_1 = \omega(a^\dagger a + b^\dagger b) + \frac{i\kappa\Delta(t)}{2}(ab - a^\dagger b^\dagger) - \frac{U\Delta(t)}{2}(a^\dagger b^\dagger + ab) \quad (11)$$

and

$$H_2 = -\omega(\tilde{a}\tilde{a}^\dagger + \tilde{b}\tilde{b}^\dagger) + \frac{i\kappa\Delta(t)}{2}(\tilde{a}\tilde{b} - \tilde{a}^\dagger\tilde{b}^\dagger) - \frac{U\Delta_1(t)}{2}(\tilde{a}^\dagger\tilde{b}^\dagger + \tilde{a}\tilde{b}). \quad (12)$$

The solution of (7) is then given by

$$|\rho(t)\rangle = (\exp[-i \int dt H_1] \otimes \exp[-i \int dt H_2])|\rho(0)\rangle, \quad (13)$$

where  $|\rho(0)\rangle$  is an initial state in  $\mathcal{H} \otimes \mathcal{H}^*$ .

It is clear from the above that the two Hamiltonians  $H_1$  and  $H_2$  are independent in the sense that  $H_1$  is acting on non-tildian system and  $H_2$  is acting on tildian system. Hence, we can work with one of the Hamiltonian and similar thing work for the other Hamiltonian (except for interchanging between  $\omega$  and  $-\omega$ ).

To study the decoherence and entanglement properties of the two-mode states under the action of the master equation (7), we would like to exploit the underlying symmetry associated with the Hamiltonians (11) and (12). This is accomplished by defining the following

$$\mathcal{N} = a^\dagger a + b^\dagger b, \quad \mathcal{K}_+ = a^\dagger b^\dagger, \quad \mathcal{K}_- = ab \quad (14)$$

$$\tilde{\mathcal{N}} = \tilde{a}\tilde{a}^\dagger + \tilde{b}\tilde{b}^\dagger, \quad \tilde{\mathcal{K}}_+ = \tilde{a}^\dagger \tilde{b}^\dagger, \quad \tilde{\mathcal{K}}_- = \tilde{a}\tilde{b} \quad (15)$$

which satisfy the  $su(1,1)$  algebra

$$[\mathcal{N}, \mathcal{K}_+] = \mathcal{K}_+, \quad [\mathcal{N}, \mathcal{K}_-] = \mathcal{K}_-, \quad [\mathcal{K}_+, \mathcal{K}_-] = 2\mathcal{N}. \quad (16)$$

Similar algebra holds for tildians. Rewriting the Hamiltonian (11) in terms of  $su(1,1)$  generators one gets

$$-iH_1 = -i\omega\mathcal{N} + \frac{\kappa\Delta(t)}{2}(\mathcal{K}_- - \mathcal{K}_+) + i\frac{U\Delta(t)}{2}(\mathcal{K}_+ + \mathcal{K}_-), \quad (17)$$

and similarly the Hamiltonian (12). It is clear that the Hamiltonians are associated with the  $su(1,1)$  symmetry. Hence, the underlying symmetry of the master equation (7) is  $su(1,1) \times su(1,1)$ . One of the important features of this  $su(1,1)$  symmetry is that it gives rise to squeezing and in turn it gives rise to entanglement. If the initial state is a two-mode Gaussian state then, under the action of two-mode squeezing, the final state turns out to be also a Gaussian state. Thus, one can use the separability criterion for Gaussian states [23] to check for the entanglement of the time evolved state.

### 3 Solution of Master Equation

To get the solution to the master equation (7) we have used the Hartree-Fock approximation and treated the approximated field as background field, and the later has to be computed self-consistently in order to get the solution of (7). For doing a self-consistent analysis we will first exactly diagonalize the Hamiltonians eqs (11) and (12) with the help of the underlying  $su(1, 1)$  symmetry. It is evident from the eqs (11) and (12) that the Hamiltonian in eq (7) is decoupled into tildien and non tildien parts as mentioned in eq (10). Hence, we work with one of the Hamiltonians and similar analysis goes through for the other Hamiltonian. By considering the Hamiltonian

$$H_1 = \omega(a^\dagger a + b^\dagger b) + \frac{i\kappa\Delta(t)}{2} (ab - a^\dagger b^\dagger) - \frac{U\Delta(t)}{2} (a^\dagger b^\dagger + ab), \quad (18)$$

and applying the following transformation

$$A = \mu a + \nu^* b^\dagger, \quad A^\dagger = \mu^* a^\dagger + \nu b \quad (19)$$

$$B = \mu b + \nu^* a^\dagger, \quad B^\dagger = \mu^* b^\dagger + \nu a \quad (20)$$

where  $\mu = e^{i\eta}|\mu|$  and  $\nu = e^{i\eta}|\nu|$ , we diagonalize the Hamiltonian  $H_1$ . Similar analysis goes through for the other Hamiltonian  $H_2$ . With a bit of algebra for  $\kappa = 0$ , one can exactly diagonalize the Hamiltonian (18) by using the transformations (19) and (20). Let  $\kappa = 0$  for  $t = 0$ , then we evolve the master equation (6) at this time. Then the corresponding non tildien Hamiltonian  $H_0$  is given by

$$H_0 = \omega(a^\dagger a + b^\dagger b) - \frac{U\Delta(t)}{2} (a^\dagger b^\dagger + ab), \quad (21)$$

and the final diagonalized Hamiltonian  $H_f$  (which is diagonalized version of  $H_0$ ), after the unitary transformation (19) and (20), is given by

$$H_f = S^{-1}(r)H_0S(r) = \Omega^2[A^\dagger A + B^\dagger B + 1] \quad (22)$$

where

$$S(r) = \exp[r\mathcal{K}_- - r^*\mathcal{K}_+] = \exp[ra^\dagger b^\dagger - r^*ab] \quad (23)$$

here  $\Omega^2 = 4\omega^2 - U^2\Delta^2(0)$  and  $r$  is related to  $\mu$  and  $\nu$  in eq (19) and (20) via the following Bogolyubov coefficients :

$$\mu = \cosh(r) = \frac{\omega}{\sqrt{\omega^2 - \frac{U^2\Delta^2(0)}{4}}}, \quad \nu = \sinh(r) = \frac{U\Delta(0)}{2\sqrt{\omega^2 - \frac{U^2\Delta^2(0)}{4}}}. \quad (24)$$

Note that the above satisfy

$$|\mu|^2 - |\nu|^2 = 1. \quad (25)$$

This fixes  $\Delta^2(0) = \frac{\omega^2+1}{U^2} = \text{Constant}$ . The solution to the Schrodinger

$$i\hbar \frac{\partial}{\partial t} |\psi_0(t)\rangle = -iH_0 |\psi_0(t)\rangle \quad (26)$$

where Hamiltonian  $H_0$  is given by eq (21), is given by

$$|\psi_0(t)\rangle = \exp[r\mathcal{K}_- - r^*\mathcal{K}_+] |\psi_0(0)\rangle = \exp[ra^\dagger b^\dagger - r^*ab] |\psi_0(0)\rangle. \quad (27)$$

Similar solution exists for the  $\tilde{H}_0$

$$\tilde{H}_0 = \omega(\tilde{a}^\dagger \tilde{a} + \tilde{b}^\dagger \tilde{b}) - \frac{U\Delta(t)}{2}(\tilde{a}^\dagger \tilde{b}^\dagger + \tilde{a}\tilde{b}). \quad (28)$$

### 3.1 Self Consistency Analysis

Now, we compute the background field  $\Delta(t)$  self-consistently by taking the initial state of the two-mode system to be the vacuum state. In thermo field dynamic notation the two-mode vacuum state is  $|\rho(0)\rangle = |00, \tilde{0}\tilde{0}\rangle$  and in the usual notation  $\rho(t) = e^{-i \int dt \hat{H}} |0, 0\rangle \langle 0, 0| e^{i \int dt \hat{H}}$ . In thermo field dynamics the background field  $\Delta(t)$  is computed as expectation value of the averaged creation and annihilation operators, and is given by

$$\Delta(t) = \langle ab \rangle = \langle I | ab | \rho(t) \rangle = \text{Tr}(ab\rho(t)). \quad (29)$$

After doing a bit of algebra (see Appendix B) one gets  $\Delta(t)$  to be

$$\Delta(t) = (1 + i\omega t)\Delta(0) + \frac{(U - i\kappa)}{2} \int_0^t dt' \Delta(t') \sinh^2(U\Delta(0)t). \quad (30)$$



This is nothing but the Fredholm integral equation of the second kind. Computing the integral upto first order one gets

$$\Delta(t) = (1 + i\omega t)\Delta(0) - \frac{(U - i\kappa)(1 - i\omega)}{2} \left[ \frac{\lambda t}{2} - \frac{\lambda}{2\Omega} \sinh(2\Omega t) \right]. \quad (31)$$

By considering  $\Omega t$  to be small (so that  $\sinh(2\Omega t) \simeq 2\Omega t$ ) one gets

$$\Delta(t) = (1 + i\omega t)\Delta(0). \quad (32)$$

Hence the solution of master equation (7) is given by eq (13), which when written in terms of  $su(1, 1)$  generators becomes

$$\begin{aligned} |\rho(t)\rangle &= \exp[-i\hat{H}'t]|\rho(0)\rangle \\ &= \left( \exp[\zeta_{a3}\mathcal{N} + \zeta_{a-}\mathcal{K}_- + \zeta_{a+}\mathcal{K}_+] \otimes \exp[\zeta_{b3}\tilde{\mathcal{N}} + \zeta_{b-}\tilde{\mathcal{K}}_- + \zeta_{b+}\tilde{\mathcal{K}}_+] \right) |\rho(0)\rangle, \end{aligned} \quad (33)$$

here  $\zeta_{a3} = i\omega t$ ,  $\zeta_{a-} = \int dt \frac{\Delta(t)}{2} (iU + \kappa)$ ,  $\zeta_{a+} = \int dt \frac{\Delta(t)}{2} (iU - \kappa)$ ,  $\zeta_{b3} = i\omega t$ ,  $\zeta_{b-} = \int dt \frac{\Delta(t)}{2} (iU + \kappa)$  and  $\zeta_{b+} = \int dt \frac{\Delta(t)}{2} (iU - \kappa)$ . Using disentanglement formula [22] one can write eq (33) as

$$\begin{aligned} |\rho(t)\rangle &= \left( \exp[\Gamma_{a+}\mathcal{K}_+] \exp[\ln(\Gamma_{a3}\mathcal{N})] \exp[\Gamma_{a-}\mathcal{K}_-] \right. \\ &\quad \left. \otimes \left( \exp[\Gamma_{b+}\tilde{\mathcal{K}}_+] \exp[\ln(\Gamma_{b3}\tilde{\mathcal{N}})] \exp[\Gamma_{b-}\tilde{\mathcal{K}}_-] \right) \right) |\rho(0)\rangle. \end{aligned} \quad (34)$$

where

$$\Gamma_{i\pm} = \frac{2\zeta_{i\pm} \sinh \phi_i}{2\phi_i \cosh \phi_i - \zeta_{i3} \sinh \phi_i}, \quad \Gamma_{i3} = \frac{1}{\left( \cosh \phi_i - \frac{\zeta_{i3}}{2\phi_i} \sinh \phi_i \right)^2} \quad (35)$$

with

$$\phi_i^2 = \frac{\zeta_{i3}^2}{4} - \zeta_{i+}\zeta_{i-} \quad (36)$$

and  $i$  stand for  $a$  and  $b$ .

It can be clearly seen that the  $\Gamma_i$  are functions of the background field  $\Delta(t)$ . By using the expression for  $\Delta(t)$  from eq (32) (which is valid for small  $t$ ) we get

$$\Gamma_{i\pm} = \frac{\Delta(0)t}{2} (iU \pm \kappa) \left( 1 + \frac{\omega^2 t^2}{4} \right). \quad (37)$$

By taking  $(iU \pm \kappa) = -\zeta e^{i\phi}$  then one gets

$$\Gamma_{i\pm} = -\frac{\Delta(0)\zeta t}{2}\left(1 + \frac{\omega^2 t^2}{4}\right)e^{\pm i\phi}. \quad (38)$$

By considering any arbitrary initial state  $\rho(0) = \sum_{m,n}^\infty \rho_{m,n}|m\rangle\langle n|$  of a single mode system where  $|m\rangle$  and  $|n\rangle$  are number of states, in the thermo field dynamic notation, there by using eq(76) (of appendix D),  $|\rho(0)\rangle$  takes the form

$$|\rho(0)\rangle = \sum_{m,n}^\infty \rho_{m,n}(0)|m,n\rangle. \quad (39)$$

Putting eq (39)in (34) one gets

$$\begin{aligned} \rho_{m,n}(t) = & \sum_{q'=0}^{\min(m',n')} \sum_{p'=0}^\infty \left[ \binom{m'+p'-q'}{p'} \binom{n'+p'-q'}{p'} \binom{m'}{q'} \binom{n'}{q'} \right]^{\frac{1}{2}} \\ & \times \sum_{q=0}^{\min(m,n)} \sum_{p=0}^\infty \left[ \binom{m+p-q}{p} \binom{n+p-q}{p} \binom{m}{q} \binom{n}{q} \right]^{\frac{1}{2}} \\ & \times [\Gamma_{a+}]^{p'} [\Gamma_{a3}]^{(m'+n'-2q'+1)/2} [\Gamma_{a-}]^{q'} [\Gamma_{b+}]^p [\Gamma_{b3}]^{(m+n-2q+1)/2} [\Gamma_{b-}]^q \\ & \times \rho_{m+p+p'-(q+q'), n+p+p'-(q+q')}(0). \end{aligned} \quad (40)$$

## 4 Entanglement

The solution of the master equation (6) is a pure state in thermo field dynamic notation and is given by

$$\begin{aligned} |\rho(t)\rangle = & (\exp[\Gamma_{a+}\mathcal{K}_+] \exp[\ln(\Gamma_{a3}\mathcal{N})] \exp[\Gamma_{a-}\mathcal{K}_-] \\ & \times \exp[\Gamma_{b+}\tilde{\mathcal{K}}_+] \exp[\ln(\Gamma_{b3}\tilde{\mathcal{N}})] \exp[\Gamma_{b-}\tilde{\mathcal{K}}_-]) |\rho(0)\rangle. \end{aligned} \quad (41)$$

where  $\Gamma_{i\pm}$  are given in eq (38). In this master equation, as there is no mixing between the tildian and non-tildian modes in the Liouville space, one can write the solution in the system

Hilbert space as

$$\begin{aligned} \rho(t) = & (\exp[\Gamma_{a+}\mathcal{K}_+]\exp[\ln(\Gamma_{a3}\mathcal{N})\exp[\Gamma_{a-}\mathcal{K}_-]|\psi(0)\rangle\langle\psi(0)|) \\ & (\exp[\Gamma_{a+}\mathcal{K}_+]\exp[\ln(\Gamma_{a3}\mathcal{N})\exp[\Gamma_{a-}\mathcal{K}_-]) \end{aligned} \quad (42)$$

where again  $\Gamma_{i\pm}$  are given in eq (38). One can clearly see that this a two-mode pure squeezed state of the two-mode Hilbert space. It is well known that two-mode squeezing gives rises to entanglement. As an example, we take the initial state to be two-mode thermal state. Then, to calculate the entanglement of the time evolved state we go over to phase space by following transformation

$$a = \frac{1}{\sqrt{2}}(x + ip_x), \quad a^\dagger = \frac{1}{\sqrt{2}}(x - ip_x), \quad (43)$$

$$b = \frac{1}{\sqrt{2}}(y + ip_y), \quad b^\dagger = \frac{1}{\sqrt{2}}(y - ip_y). \quad (44)$$

Putting them in the eq (42) one gets the two-mode squeezed thermal state in the phase space corresponding to the two-mode the real squeezing transformation

$$S(r) = \begin{pmatrix} \cosh(r) & 0 & \sinh(r) & 0 \\ 0 & \cosh(r) & 0 & -\sinh(r) \\ \sinh(r) & 0 & \cosh(r) & 0 \\ 0 & -\sinh(r) & 0 & \cosh(r) \end{pmatrix}, \quad (45)$$

where the squeezing parameter is given by  $r = \frac{\Delta(0)}{2}(1 + \frac{\omega^2 t^2}{4})\zeta t$ . The covariance matrix of a two-mode thermal state is given by

$$\sigma = n_1 1 \oplus n_2 1 \quad (46)$$

where the  $n_1$  and  $n_2$  are symplectic eigenvalues. The covariance matrix for the state  $\rho(t)$  in eq (42) is given by

$$V = S(r)\sigma S^\dagger(r) \quad (47)$$

where

$$V = \begin{pmatrix} p & 0 & -s & 0 \\ 0 & p & 0 & s \\ -s & 0 & q & 0 \\ 0 & s & 0 & q \end{pmatrix} \quad (48)$$

with  $p = n_1 \cosh^2(r) + n_2 \sinh^2(r)$ ,  $q = n_1 \sinh^2(r) + n_2 \cosh^2(r)$  and  $s = \pm \frac{n_1 + n_2}{2} \sinh(2r)$ .

So the covariance matrix is of the form

$$V = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix} \quad (49)$$

where

$$A = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}, B = \begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix}, C = \begin{pmatrix} -s & 0 \\ 0 & s \end{pmatrix}, J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (50)$$

Then the separability condition [23] for any two-mode state reads as

$$\det A \det B + \left( \frac{1}{4} - |\det C| \right)^2 - \text{tr}(AJCJBJC^T J) \geq \frac{1}{4}(\det A + \det B) \quad (51)$$

So, according to condition (51) in eq (42), the state  $\rho(t)$  is entangled if and only if the condition is satisfied. Then the thermal state is entangled

$$\sinh^2(r) \geq \frac{(n_2^2 - 1)(n_1^2 - 1)}{(n_1 + n_2)^2}. \quad (52)$$

For  $n_1 = n_2 = n$ , the amount of entanglement in  $\rho(t)$  is given in terms of logarithmic negativity

$$E_N(r) = -\frac{1}{2}[\text{Log}(e^{-4r}/n)]. \quad (53)$$

## 5 Decoherence

As we have seen the previous section, that the solution to  $\rho(t)$  of the master equation (6) in the Hilbert space  $H$  is given by

$$\begin{aligned} \rho(t) = & (\exp[\Gamma_{a+}\mathcal{K}_+] \exp[\ln(\Gamma_{a3}\mathcal{N}) \exp[\Gamma_{a-}\mathcal{K}_-]] |\psi(0)\rangle \langle \psi(0)| \\ & (\exp[\Gamma_{a+}\mathcal{K}_+] \exp[\ln(\Gamma_{a3}\mathcal{N}) \exp[\Gamma_{a-}\mathcal{K}_-]] \end{aligned} \quad (54)$$

where again  $\Gamma_{i\pm}$  are given in eq (38). To calculate decoherence effects of  $\rho(t)$  we compute  $\rho^2$  and is given as follows note that

$$\rho(t) = \exp\left[-\Delta(0)2\left(1 + \frac{\omega^2 t^2}{4}\right)\zeta t\right]\rho(0) \quad (55)$$

Then

$$\text{Tr}[\rho^2(t)] = \text{Tr}\left[\sum_{m,n} \langle m, n | \rho^2(t) | m, n \rangle\right] = \exp\left[-\Delta(0)2\left(1 + \frac{\omega^2 t^2}{4}\right)\zeta t\right] \quad (56)$$

The behaviour of decoherence is plotted fig. 2. One can see immediately that for the short time it self as the value of damping coefficient increases the system decoheres faster.

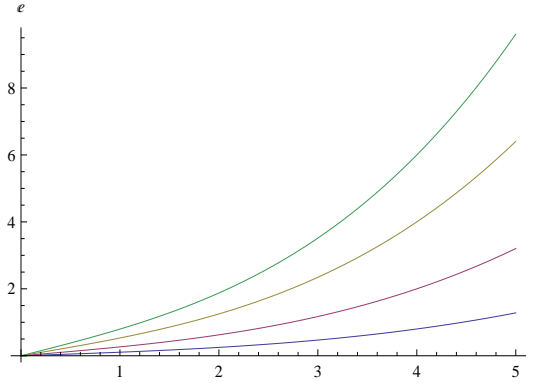


Figure 1: Time vs Entanglement: Here  $\omega = .25$ ,  $\Delta(0) = .25$  and  $n = .5$  the blue, pink, orange and green corresponds to  $\zeta = 0.1, 0.25, 0.5$  and  $0.75$ .

## 6 Conclusion

In this paper, the techniques of thermo field dynamics and the Hartree-Fock approximation are used to solve the master equation for the special case of two-mode Bose-Hubbard model in the presence of a non-linear damping. We have treated the approximated field self-consistently and computed it for a small time  $t$  analytically. Then, the decoherence and the

entanglement for this system are computed. We show that the entanglement for this system for a short time increases when the initial state is in two-mode thermal state. We interpret this behaviour due to the existence of the non-linear medium. To get the exact picture for the behaviour of the entanglement for a long time, one has to do the numerical studies. It can be seen from the decoherence plot that as the value of the damping coefficient increases the damping in the system is faster, as expected. We expect that the further numerical studies using this model will give better results and these results can be applied to condensed matter systems.

## 7 Appendix

### 7.1 Appendix A: Calculation of Bogolyubov Coefficients

Consider the following Hamiltonian

$$H_0 = \omega(a^\dagger a + b^\dagger b) - \frac{U\Delta(t)}{2}(a^\dagger b^\dagger + ab), \quad (57)$$

By applying the following Bogolyubov or squeezing transformation

$$\begin{aligned} A &= \mu a + \nu^* b^\dagger, \quad A^\dagger = \mu^* a^\dagger + \nu b \\ B &= \mu b + \nu^* a^\dagger, \quad B^\dagger = \mu^* b^\dagger + \nu a \end{aligned}$$

(where  $\mu = e^{i\eta}|\mu|$  and  $\nu = e^{i\eta}|\nu|$ ), we diagonalize the Hamiltonian  $H_0$ . Let us first rewrite the Hamiltonian in a convenient form as

$$H_0 = [a^\dagger \quad b] \begin{pmatrix} m & n \\ n & m \end{pmatrix} \begin{bmatrix} a \\ b^\dagger \end{bmatrix} \quad (58)$$

where  $m = i\omega$  and  $n = \frac{-iU\Delta(0)}{2}$ . By identifying the coefficient matrix appearing in the eq (58) with

$$\begin{pmatrix} m & n \\ n & m \end{pmatrix} = \begin{pmatrix} \cosh(r) & \sinh(r) \\ \sinh(r) & \cosh(r) \end{pmatrix}, \quad (59)$$

and with a suitable normalization  $\sqrt{|m|^2 - |n|^2}$ , the Bogolyubov coefficients can be read off as follows

$$\mu = \cosh(r) = \frac{\omega}{\sqrt{\omega^2 - \frac{U^2 \Delta^2(0)}{4}}}, \quad \nu = \sinh(r) = \frac{U \Delta(0)}{2\sqrt{\omega^2 - \frac{U^2 \Delta^2(0)}{4}}}. \quad (60)$$

The above satisfy

$$|\mu|^2 - |\nu|^2 = 1. \quad (61)$$

## 7.2 Appendix B: Computation of $\Delta(t)$

Consider the following field variable appeared in eq (11) and (12)

$$\Delta(t) = \langle ab \rangle \quad (62)$$

The initial state being the vacuum state the thermo field initial state is given by  $|\rho(0) = |0, 0, \tilde{0}\tilde{0}\rangle$  and in usual notation

$\rho(t) = e^{-i \int dt H} |0, 0\rangle \langle 0, 0| e^{i \int dt H}$  where  $H$  is equal to  $H_1$  in eq (11). So we have

$$\begin{aligned} \langle ab \rangle &= \langle I | ab | \rho(t) \rangle = \text{Tr}(ab \rho(t)) = \text{Tr}(ab e^{-i \int dt H} |0, 0\rangle \langle 0, 0| e^{i \int dt H}) \\ &= \sum_{m,n} \langle m, n | ab e^{-i \int dt H} |0, 0\rangle \langle 0, 0| e^{i \int dt H} |m, n\rangle \\ &= \sum_{m,n} \langle 0, 0 | e^{i \int dt H} |m, n\rangle \langle m, n | ab e^{-i \int dt H} |0, 0\rangle \\ &= \langle 0, 0 | e^{i \int dt H} \left( \sum_{m,n} |m, n\rangle \langle m, n| \right) ab e^{-i \int dt H} |0, 0\rangle \\ &= \langle 0, 0 | e^{i \int dt H} ab e^{-i \int dt H} |0, 0\rangle \\ &= \langle 0, 0 | (1 + i \int dt H) ab (1 - i \int dt H) |0, 0\rangle \\ &= \langle 0, 0 | (ab + i \int dt [H, ab]) |0, 0\rangle \\ &= \langle 0, 0 | ab \rangle + i \int dt \langle 0, 0 | [H, ab] |0, 0\rangle \\ &= \mu\nu + i \int dt \langle 0, 0 | [\zeta_{a3} \mathcal{N} + \zeta_{a-} \mathcal{K}_- + \zeta_{a+} \mathcal{K}_+, \mathcal{K}_-] |0, 0\rangle \\ &= \mu\nu + i \int dt \omega \mu\nu + \int dt \zeta_{a3} |\mu|^2 \end{aligned}$$

$$\begin{aligned}
&= (1 + i\omega t)\Delta(0) + \frac{(iU - \kappa)}{2} \int dt \Delta(t) \sinh^2(r) \\
&= (1 + i\omega t)\Delta(0) + \frac{(iU - \kappa)}{2} \int dt \Delta(t) \sinh^2(U\Delta(0)t).
\end{aligned} \tag{63}$$

### 7.3 Appendix C: Calculation of $\phi$ and $\Gamma$

Considering only for small time  $t$ , i.e., neglecting  $\Delta^2(t)$  and higher order terms one has from (36)

$$\phi_i^2 = \frac{\zeta_{i3}^2}{4} = -\frac{\omega^2 t^2}{4}, \tag{64}$$

which gives

$$\Gamma_{i\pm} = \frac{2\zeta_{i\pm} \sinh(\frac{i\omega t}{2})}{i\omega t (\cosh(\frac{i\omega t}{2}) + \sinh(\frac{i\omega t}{2}))} = \frac{2\zeta_{i\pm} \sin(\frac{\omega t}{2})}{\omega t} e^{-\frac{i\omega t}{2}}. \tag{65}$$

Here we have used  $\cosh(x) + \sinh(x) = e^x$  and  $-i\sinh(ix) = \sin(x)$ . Again considering  $\omega t$  to be small, that is using  $\sin(\omega t) = \omega t$  one has from eq(65):

$$\Gamma_{i\pm} = \zeta_{i\pm} (1 - \frac{i\omega t}{2}) \tag{66}$$

Thus

$$\Gamma_{i\pm} = \int dt \frac{\Delta(t)}{2} (iU \pm \kappa) (1 - \frac{i\omega t}{2}). \tag{67}$$

Then putting the value of  $\Delta(t)$  from eq (32) into eq(67) one gets

$$\Gamma_{i\pm} = \int_0^t dt' \frac{\Delta(t')}{2} (iU \pm \kappa) (1 - \frac{i\omega t}{2}) = \int_0^t dt' \frac{(1 + i\omega t')\Delta(0)}{2} (iU \pm \kappa) (1 - \frac{i\omega t}{2}) \tag{68}$$

$$= \frac{\Delta(0)}{2} (iU \pm \kappa) (t + i\omega \frac{t^2}{2}) (1 - \frac{i\omega t}{2}) = \frac{\Delta(0)t}{2} (iU \pm \kappa) (1 + \frac{\omega^2 t^2}{4}). \tag{69}$$

### 7.4 Appendix D: Thermo Field Dynamics

A brief description of TFD is given below. The dissipative term in any master equations makes it difficult to apply the usual Schroedinger equation techniques with pure states to mixed states. The thermo field dynamics (TFD) provides such a formalism. In TFD the



mixed state averages are expressed as scalar products and the dynamics is given in terms of Schroedinger like equation. A density operator  $\rho = |N\rangle\langle N|$  corresponding to a Fock state  $|N\rangle$  in the Hilbert space  $\mathcal{H}$  is viewed in TFD as a vector  $\rho = |N, \tilde{N}\rangle$  in an extended Hilbert space  $\mathcal{H} \otimes \mathcal{H}^*$ . The central idea in TFD is to construct a density operator  $|\rho^\alpha\rangle, 1/2 \leq \alpha \leq 1$  as a vector in the extended Hilbert space  $\mathcal{H} \otimes \mathcal{H}^*$ .

Here the averages of operators with respect to  $\rho$  reduces to a scalar product:

$$\langle A \rangle = \text{Tr}[A\rho] = \langle \rho^{1-\alpha} | A | \rho^\alpha \rangle, \quad (70)$$

where  $|\rho^\alpha\rangle$  is given by

$$|\rho^\alpha\rangle = \hat{\rho}^\alpha |I\rangle, \text{ with, } \hat{\rho}^\alpha = \rho^\alpha \otimes I, \quad (71)$$

where  $|I\rangle$  is the resolution of the identity

$$|I\rangle = \sum |n\rangle\langle n| = \sum |n\rangle \otimes |\tilde{n}\rangle \equiv \sum |n, \tilde{n}\rangle, \quad (72)$$

in terms of a complete orthonormal basis  $\{|n\rangle\}_{n=0}^\infty$  in  $\mathcal{H}$ . The state vector  $|I\rangle$  takes a normalized vector to another normalized vector in the extended Hilbert space  $\mathcal{H} \otimes \mathcal{H}^*$ . The matrix  $A(a, a^\dagger)$  acts like  $A \otimes I$ .

(It may be noted that for any density operator the states  $|\rho^\alpha\rangle, 1/2 \leq \alpha \leq 1$  have a finite norm in the extended Hilbert space  $\mathcal{H} \otimes \mathcal{H}^*$ . This is not in general true for the state  $|\rho^{1-\alpha}\rangle, 1/2 \leq \alpha \leq 1$ , which includes  $|I\rangle$ . These state are regarded as formal but extremely useful constructs.)

The creation and the annihilation operators  $a^\dagger, \tilde{a}^\dagger, a$ , and  $\tilde{a}$  are introduced as follows

$$a|n, \tilde{m}\rangle = \sqrt{n}|n-1, \tilde{m}\rangle, \quad a^\dagger|n, \tilde{m}\rangle = \sqrt{n+1}|n+1, \tilde{m}\rangle, \quad (73)$$

$$\tilde{a}|n, \tilde{m}\rangle = \sqrt{\tilde{m}}|n, \tilde{m}-1\rangle, \quad \tilde{a}^\dagger|n, \tilde{m}\rangle = \sqrt{\tilde{m}+1}|n, \tilde{m}+1\rangle. \quad (74)$$

The operators  $a$  and  $a^\dagger$  commute with  $\tilde{a}$  and  $\tilde{a}^\dagger$ . It is clear from the above that  $a$  acts on the vector space  $\mathcal{H}$  and  $\tilde{a}$  acts on vector space  $\mathcal{H}^*$ . From the expression for  $|I\rangle$  in terms of the number states

$$|I\rangle = \sum_n |n, \tilde{n}\rangle, \quad (75)$$

it follows that

$$a|I\rangle = \tilde{a}^\dagger|I\rangle, \quad a^\dagger|I\rangle = \tilde{a}|I\rangle, \quad (76)$$

and hence for any operator  $A$  (written in terms of  $a$ ,  $a^\dagger$  and their complex conjugates), one has

$$A|I\rangle = \tilde{A}^\dagger|I\rangle, \quad (77)$$

where  $\tilde{A}$  is obtained from  $A$  by making the replacements tilde conjugation rules  $a \rightarrow \tilde{a}$ ,  $a^\dagger \rightarrow \tilde{a}^\dagger$ ,  $\alpha \rightarrow \alpha^*$ . An immediate consequence of this is that the state  $|\rho^\alpha\rangle$  which remains unchanged under the replacements  $a \rightarrow \tilde{a}$ ,  $a^\dagger \rightarrow \tilde{a}^\dagger$ , and c number  $\rightarrow$  complex conjugates  $C$  by applying the hermiticity property of  $\rho$  i.e.  $\rho^\dagger = \rho$ . The tildian property reflects the hermiticity property of the density operator.

The evolution of a conservative system in terms of  $\rho^\alpha$  is given by the von Neumann equation

$$\frac{\partial}{\partial t}\rho^\alpha(t) = \frac{-i}{\hbar}[H, \rho^\alpha], \quad (78)$$

by applying  $|I\rangle$  from the right and one gets

$$\frac{\partial}{\partial t}|\rho^\alpha(t)\rangle = -i\hat{H}|\rho^\alpha\rangle, \quad (79)$$

where

$$-i\hat{H} = i(H - \tilde{H}). \quad (80)$$

In TFD, one can derive a Schroedinger like equation for any state  $|\rho^\alpha\rangle$  with arbitrary value of  $\alpha$ . For dissipative systems the evolution equation is given by master equation

$$\frac{\partial}{\partial t}\rho(t) = \frac{-i}{\hbar}(H\rho - \rho H) + L\rho, \quad (81)$$

where  $L$  is Liouville term. The non equilibrium thermo field dynamics is developed and analysed in  $\alpha = 1$  representation. Hence, from now on, we work in,  $\alpha = 1$  representation [19, 20, 18]. In this representation, for any hermitian operator  $A$ , one has

$$\langle A \rangle = \langle I|A|\rho \rangle = \langle A|\rho \rangle = Tr(A\rho). \quad (82)$$

By applying  $|I\rangle$  to the eq (81) from the right one goes over to TFD and the master equation is given by (2). with  $-i\hat{H}$  being a tildian and thus problem of solving master equation is reduced to solving a Schroedinger like equation namely eq (2).

Historically the thermo field dynamics was developed in  $\alpha = \frac{1}{2}$  representation. In this representation  $|\rho_0^{\frac{1}{2}}\rangle$  is related to the  $|0,0\rangle$  by a unitary transformation, which is nothing but the Caves-Schumaker state, for details ref [9, 10, 11, 12, 13, 18, 21].

## References

- [1] I. Bloch, Nature **453**, 1016, (2008).
- [2] L. O. Castanos and R. Jauregui Phys. Rev. **A 82**, 053815, (2010).
- [3] R. V. Mishmash and L. D. Carr, Phys. Rev. Lett. **103**, 140403 (2009)
- [4] F Benatti, R Floreanini and U Marzolino, J. Phys. B: At. Mol. Opt. Phys. **44**, 091001, (2011).
- [5] K. Pawłowski and K. Rzazewski, Phys. Rev. **A 81**, 013620, (2010)
- [6] K. Pawłowski, P. Zin, K. Rzazewski, M. Trippenbach, Phys. Rev. **A 83**, 033606, (2011)
- [7] D. Jaksch, C. Bruder, J. I. Cirac, C. W. Gardiner, and P. Zoller Phys. Rev. Lett. **81**, 3108, (1998).
- [8] P. Wurtz, T. Langen, T. Gericke, A. Koglbauer, and H. Ott Phys. Rev. Lett **103**, 080404, (2009).
- [9] L. Laplae, F. Mancini and H, Umezawa, Phys. Rev. **C10**, 151, (1974)
- [10] Y. Takahashi and H. Umezawa Collect. Phenom. **2**, 55 (1975); reprinted in Int. J. Mod. Phys. **B 10**, 1996, 1755, (1996).

- [11] I. Ojima, Ann. Phys. **137**, 1 (1981)
- [12] H. Umezawa, H. Matsumoto and M. Tachiki, *Thermo field Dynamics and Condensed States* (North Holland, Amsterdam, 1982)
- [13] H. Umezawa, Proceedings of the conference on Thermofield Dynamics, Banf, Canada 1993.
- [14] S. Chaturvedi and V. Srinivasan, *J. Mod. Opt.* **38**, 777, (1991).
- [15] S. Chaturvedi and V. Srinivasan, Phy Rev **A 43**, 4054, (1991).
- [16] H.-Y. Fan, H.-L Lu and Y. Fan, Ann. Phys. **321**,, 480, (2006).
- [17] A. Wünsche, J. Opt. B: Quantum Semiclass. Opt. **1**, (1999) .
- [18] P. Shanta, S. Chaturvedi, V. Srinivasan and A.K. Kapoor, Int. J. Mod. Phys. **B 10**, 1573,(1996).
- [19] S. Chaturvedi, V. Srinivasan, G. S. Agarwal, J Phy **A 32** 1909, (1999).
- [20] P. Shanta, S. Chaturvedi and V. Srinivasan, *Mod Phy Lett A* **72**, 2381, (1996).
- [21] S. Chaturvedi, R. Sandhya, R. Simon and V. Srinivasan, Phy Rev **A 41**, 3969, (1989).
- [22] A. M. Perelomov, *Generalized Coherent States and their Applications*, (Springer- Verlag, Berlin, 1986).
- [23] R. Simon 2000, Phy Rev Lett **84**, 2726.
- [24] L. M. Duan, G. Giedke, J. I. Cirac, P. Zoller, Phys. Rev. Lett. **84**, 2722 (2000)
- [25] P. Shanta, S. Chaturvedi, V. Srinivasan, G.S. Agarwal and C. L.Metha, Phy Rev Lett **72**, 1447, (1994).

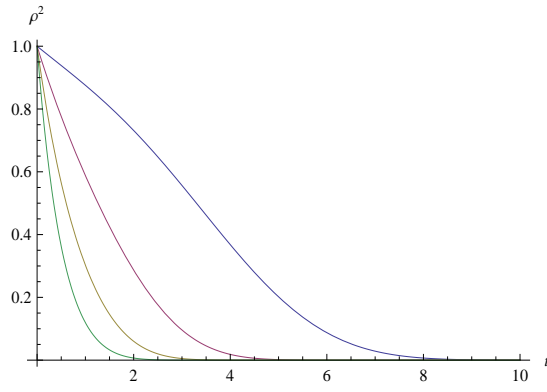


Figure 2: Time vs Decoherence: Here  $\omega = .25$   $\Delta(0) = .25$  the blue, pink, orange and green corresponds to  $\zeta = 1, 0.75, 0.5$  and  $0.25$ .